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Matching and Information Design in Marketplaces*

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February 3, 2023

Abstract

There are many markets that are networked in these sense that not all consumers have access to (or are aware of) all products, while, at the same time, firms have some information about consumers and can distinguish some consumers from some others (for example, in online markets through cookies). With unit demand and price-setting firms we give a complete characterization of all welfare outcomes achievable in equilibrium (for arbitrary buyer-seller networks and arbitrary information structures), as well as the designs (networks and information structures) which implement them.

1 Introduction

Many markets are networked in the sense that not all consumers can access all products. It may be the case that consumers are not aware of all products, or some products might be available in certain geographic areas while others are not. There is much work exploring markets like these in the context of the literature on consumer search, the literature on networked markets or in the context of platforms that have some control over which products consumers can access. At the same time, especially in the modern economy, firms often have some information about at least some consumers enabling them to price differentially to them. This might obtain through targeted promotional campaigns to make certain consumers aware of an offer, the targeted distribution of coupons (or coupon codes), concessions being offered for certain consumers based on observable (e.g., reduced-price childrens' tickets), and so on. In this paper we ask and provide a complete answer to a basic question that does not require taking a view on how the buyer-seller network or information is determined and what microeconomic process underlie them: For the case in which consumers have unit demand

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and firms compete on prices, we characterise what combinations of producer and consumer surplus can be obtained.

This characterization extends the canonical model and results of Bergemann, Brooks, and Morris (2015) from that of monopoly to oligopolistic competition, and from all firms being able to access all consumers to any network structure.[1] Furthermore, in the modern economy interactions between consumers and firms are often mediated by an intermediary. The intermediary organizes trade in the marketplace, fulfilling the role of matchmaker—selectively matching firms' offerings to consumers—as well as the role of information provider—selectively disclosing information about consumers' preferences to sellers. An important application of our analysis is to such situations.

For example, as of 2022, there were almost 10 million third-party sellers on Amazon, most of which are small-to-medium businesses and operated exclusively via Amazon. There is considerable opacity in what determines which products are recommended to users (e.g., "Amazon's choice"), and in how search results are ranked.[2] Through the design of search, ranking and recommendation algorithms, Amazon guides consumers towards specific products and shapes the intensity of price competition and, therefore, it influences market outcomes, (see Lee and Leon (2021) for an empirical study on Amazon marketplace). Similarly, an intermediary like Google sells precise information on consumers' preferences to competing downstream firms via targeted advertising services and selects how to present search outcomes to consumers, influencing the set of firms the consumers consider buying from. The recent European Commission (EC) antitrust case against Google for self-preferencing—in effect, displaying its own services prominently so that it would be in consumers' consideration sets while hiding that of its competitors—speaks directly to the potential distortion that the ability of designing matching between firms and consumers can create.[3]

In regulating how a platform matches consumers with third-parties, it would seem necessary to first have a strategic framework that models how the platform matches consumers with firms and what information about consumer valuations is given to firms. Our paper develops this strategic framework and fully characterizes the set of achievable welfare outcomes.

We model a marketplace as a set of consumers with heterogeneous preferences over differentiated products offered by firms who compete in prices. Firms have restricted access to consumers, which can be represented by a bi-partite network (or many-to-many matching), and information about consumers' types which can potentially be very coarse, very fine-grained or anything in between. This can be thought of as being determined (or influenced)

[1] This also extends the setting of our prior work Elliott, Galeotti, Koh, and Li (2021) from all firms being able to access all consumers to any network structure.

[2] See, for example, `https://www.wired.com/story/what-does-amazons-choice-mean/`

[3] In 2017, Google was fined $2.7Bn by the EC for giving prominent placements to its own shopping service results while demoting the placement of competitor comparison shopping services. Similar opinions and cases are discussed in US, e.g., see Khan (2016).

by a platform. Regardless, given the information they have firms simultaneously set prices to the consumers they can access and consumers decide which, of the products available to them, to buy.

Our main result is a complete characterisation of the feasible surplus set—all pairs of producer and consumer surplus which can be implemented in some (Bayes correlated) equilibrium induced by designs over matching and information (Theorem 1); we also pin down the designs which can implement each point in this surplus set.

We then consider a setting in which the matching is restricted so that each consumer can access a minimum number of firms. In a reduced form, this constraint can be given a variety of interpretations. For example, it might capture the search behavior of consumers, or it could represent regulatory requirements imposed on a platform. We show that in marketplaces where the number of firms is sufficiently large, these restrictions are inconsequential, and do not alter the feasible surplus set (Proposition 2).

1.1 Related literature. A recent literature studies how the informational environment interacts with consumer and firm surplus. The focus has been primarily on the design of information under the assumption that consumers and firms are frictionlessly matched with each other. Bergemann, Brooks, and Morris (2015) studies price discrimination when a monopolist obtains additional information about consumer valuations; Elliott, Galeotti, Koh, and Li (2021) (henceforth, EGKL) extend the framework to allow for downstream competition.[4] A complementary literature has focused on buyer-seller networks, e.g., Kranton and Minehart (2001), Manea (2011) and Elliott (2015). In this work the network defines who can trade with whom and the focus is to understand how the network structure affects market outcomes under centralised or decentralised trading protocols.

We build on these two literatures by combining information design with matching/network design. The distinction between these two tools has been pointed out in the review article of Bergemann and Bonatti (2019) who build on a classification first introduced by a Federal Trade Commission report from 2014. Yet, the literature has focused mainly on either information design or network design.[5]

Recent works that share a similar spirit to our paper are Bergemann and Bonatti (2022) and Condorelli and Szentes (2022). Bergemann and Bonatti (2022) study how specific information

[4]Ali, Lewis, and Vasserman (2020) consider the case where the information that firms have is disclosed directly by consumers. Roesler and Szentes (2017) and Condorelli and Szentes (2020) study the problem in which consumers, rather than a monopoly, have uncertain valuation; Armstrong and Zhou (2022) extend this setting to the duopoly case.

[5]For the latter, several papers study the matching between consumers and firms in which consumer search is explicitly modeled (Hagiu and Jullien, 2011; Eliaz and Spiegler, 2011; De Corniere, 2016). We abstract away from modelling search to focus on the joint role of matching and information in shaping market outcomes.

structures about consumer preferences (e.g., full information, rank-order etc.) affect competition when (i) access to the consumers is auctioned off by the platform to competing firms; and (ii) firms set a menus of offline prices which serve as outside options for the consumer. We abstract away from how matching consumers and firms can be monetized (e.g., through auctions) but our approach has the advantage of yielding a precise characterization of what is achievable. Condorelli and Szentes (2022) study a setting where a platform mediates interactions between firms, each selling one unit of a differentiated product, and consumers. The platform knows consumers' valuations and designs one-to-one matches between the two-sides of the market; firms, then, make a take-it-or-leave-it offer to the matched buyer. The focus of Condorelli and Szentes (2022) is to understand how matching design may reveal information to firms about consumers' valuations and its implication for buyer-optimal matching. We focus on how the joint use of both matching and information designs shapes price competition.

2 Model

Consider a marketplace where there are n firms indexed $\mathcal{N} = \{1, 2, \dots, n\}$ each producing a differentiated product at zero marginal cost. A single consumer has unit demand and her valuation is given by $\boldsymbol{\theta} = (\theta_1, \dots, \theta_n)$ where θ_i is the consumer's valuation for firm i's product; $\boldsymbol{\theta}$ is distributed on $[0,1]^n$ according to measure μ which we will assume admits density f with full support on $[0,1]^n$. An alternative interpretation is that there is a continuum of consumers and $f(\boldsymbol{\theta})$ is the density of consumers with valuation $\boldsymbol{\theta}$. The subset of consumer types whose favourite product is that of firm i is:

$$E_i := \Big\{\boldsymbol{\theta} \in [0,1]^n : i \in \operatorname*{argmax}_{j \in \mathcal{N}} \theta_j\Big\}.$$

Since $\bigcup_{i \in \mathcal{N}} E_i = [0,1]^n$ and, since f admits a density, $\bigcap_{i \in S} E_i$ is zero measure for all non-singleton $S \subseteq \mathcal{N}$.

A design of the marketplace is a map from the consumer's types to a joint distribution over the firms that the consumer receives an offer from, and the information each firm receives about the consumer's type. Formally, a design is a map

$$\psi : [0,1]^n \to \Delta\Big([0,1]^n \times 2^{\mathcal{N}}\Big).$$

For a given realization of consumer preferences $\boldsymbol{\theta} \in [0,1]^n$, the design ψ induces a distribution (measure) $\psi(\cdot|\boldsymbol{\theta}) \in \Delta([0,1]^n \times 2^{\mathcal{N}})$ over both:

- Vector-valued messages $\boldsymbol{m} \in [0,1]^n$, where m_i is the private message to firm i about the consumer's type;

- Sets $S \in 2^{\mathcal{N}}$, where set S lists the firms that have been matched to (and hence have access to) consumer type $\boldsymbol{\theta}$; sometime we shall refer to S as the consideration set of consumer type $\boldsymbol{\theta}$.

Denote the set of possible designs with $\boldsymbol{\Psi}$.

Note that a design ψ induces a matching scheme, denoted by $\phi : \Theta \to \Delta(2^{\mathcal{N}})$, which maps consumer types into a probability distribution over consideration sets; while $\phi(S|\boldsymbol{\theta}) \in [0,1]$ denotes the probability that firms S are the ones with access to a consumer of type $\boldsymbol{\theta}$. It will sometimes be helpful to work with a specific matching scheme ϕ, in which case we are restricting the space of designs to

$$\boldsymbol{\Psi}_{\phi} := \left\{ \psi \in \boldsymbol{\Psi} : \psi([0,1]^n \times \{S\}|\boldsymbol{\theta}) = \phi(S|\boldsymbol{\theta}) \text{ for almost all } \boldsymbol{\theta} \in \Theta \text{ and all } S \in 2^{\mathcal{N}} \right\},$$

i.e., the set of designs that induce the same marginal distribution over consideration sets as matching scheme ϕ. In such cases we use $\psi_{\phi} \in \boldsymbol{\Psi}_{\phi}$ to represent the design that induced ϕ.

Given design $\psi \in \boldsymbol{\Psi}$, the timing of the game is as follows:

1. $\boldsymbol{\theta}$ is drawn from μ and observed by both the consumer and designer;
2. $(\boldsymbol{m}, S)$ is drawn from $\psi(\cdot|\boldsymbol{\theta})$ and m_i is sent privately to each firm $i \in \mathcal{N}$;
3. Firms simultaneously set prices $\boldsymbol{p} = (p_i)_{i=1}^n$;
4. The consumer of type $\boldsymbol{\theta}$ observes offers from all firms in S and chooses the offer maximizing her net utility, as long as it is not worse than her exogenous outside option, assumed to be zero.

The design selected induces a simultaneous price setting game among the firms and we focus on Bayes-Correlated Equilibria (Bergemann and Morris, 2016), henceforth equilibria. We wish to characterise the feasible surplus set which is defined as follows:

Definition 1. The feasible surplus set $SUR \subset \mathbb{R}^2_{\geq 0}$ is the pairs of producer surplus (PS) and consumer surplus (CS) that can be implemented as an equilibrium outcome of some design $\psi \in \boldsymbol{\Psi}$. The lower envelope of SUR, denoted by LE, is the set of all pairs $(CS, PS) \in SUR$ with the property that if $CS' = CS$ and $PS' < PS$ then $(CS', PS') \notin SUR$.

3 Characterization

Let $TS := \sum_{i=1}^n \int_{E_i} \theta_i f(\boldsymbol{\theta}) d\boldsymbol{\theta}$ be the total surplus. We define surplus points useful for our characterization, stated in Theorem 1.

Definition 2.

- The producer-optimal point (PO) is $(0, TS)$ i.e., it allocates total surplus to the producers.
- The consumer optimal point (CO) is a point in set SUR with the highest consumer surplus among all points in SUR. If there are multiple such points, we choose the one with highest producer surplus.
- The ϕ-consumer optimal design is a design in $\boldsymbol{\Psi}_{\phi}$ with an associated equilibrium that gives the highest consumer surplus among all designs in $\boldsymbol{\Psi}_{\phi}$ and all associated equilibria.

Theorem 1. The following characterizes SUR:

(i) The producer-optimal point (PO) is obtained through the singleton matching $\phi(\{i\}|\boldsymbol{\theta}) = 1$ for all $\boldsymbol{\theta} \in E_i$ and all $i \in \mathcal{N}$ and the information design that fully reveals consumer types.

(ii) Each point (PS, CS) in the lower envelope of SUR (LE) can be implemented through a ϕ-consumer-optimal design for some ϕ.

(iii) The consumer-optimal point (CO) is obtained through the unrestricted matching—$\phi^*(\mathcal{N}|\boldsymbol{\theta})) = 1$ for all $\boldsymbol{\theta} \in [0, 1]^n$ paired with the ϕ^*-consumer optimal design.

(iv) The feasible surplus set is the convex hull generated by the producer-optimal point PO and the lower envelope of SUR:

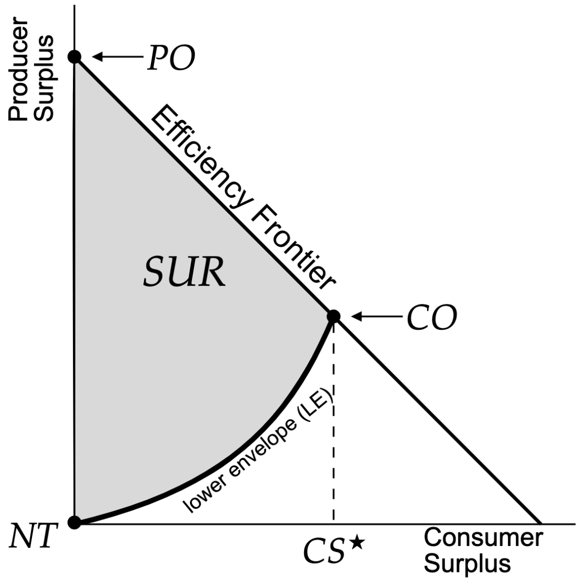
$$SUR = \text{conv}(PO \cup LE).$$

We prove Theorem 1 in Section 3.2. The proof provides further insights on designs that implement different points in SUR. Figure 1 illustrates Theorem 1. We now provide an explicit construction of SUR for the monopoly case and contrast the feasible outcomes with those obtained when the firm always has access to all consumers as studied in Bergemann, Brooks, and Morris (2015).

3.1 Construction of SUR in the monopoly case. Consider the case of a single firm and let the consumer valuation θ be distributed over $[0, 1]$ and with density f. Assume f has full support on $[0, 1]$ and is differentiable. Let p^* be the largest optimal uniform price for the monopoly and $\bar{\pi}$ the associated profit.

Bergemann, Brooks, and Morris (2015) study the surplus points that can be obtained as we vary the information that the firm can have about the consumer value, under the assumption

Figure 1: Illustration of *SUR*



that the firm has always access to the consumer. This set is depicted by the grey triangle in panel (a) of Figure 2 for the case of the uniform distribution. Regardless of the information the firm has, she can always obtain profit $\bar{\pi}$ (1/4 in the uniform case) by setting p^*; this is often inefficient as some low value consumers do not trade. If the firm perfectly learns the consumer value, the firm profit equals total surplus TS (1/2 in the uniform case). An important result of Bergemann, Brooks, and Morris (2015) is that there exists a consumer-optimal segmentation such that (i) the monopolist's profits from price discrimination is exactly $\bar{\pi}$; and (ii) the allocation is efficient hence consumer surplus is $CS^* = TS - \bar{\pi}$ (1/4 in the uniform case).

Theorem 1 generalizes this by supposing that some measure of consumers have limited access to the monopolist's offer. With a slight abuse of notation, we let $\phi(\theta)$ denote the probability that a consumer of valuation θ is matched to the firm. A few observations are in order. First, we can pick $\phi(\theta) = 1$ for all θ to recover the surplus triangle of Bergemann, Brooks, and Morris (2015). Second, we can pick $\phi(\theta) = 0$ for all θ to obtain the outcome in which consumer and producer surplus are both zero (no trade). Third, we can obtain the convex hull of the surplus triangle and the no-trade surplus point through suitable randomization. Can we obtain more surplus points than this?

To answer this question we need to characterize the lower envelope of SUR. By part (ii) of Theorem 1, each point on the lower envelope is given by the ϕ-consumer-optimal design of some matching ϕ. In the monopoly case, the ϕ-consumer-optimal design is just the consumer optimal segmentation of Bergemann, Brooks, and Morris (2015) applied to value distribution modified by matching ϕ ((unnormalized) density given by $\phi(\theta)$ instead of one). Proposition 1 characterizes the matching scheme that implements the point on the lower envelope with a profit $\pi \in (0, \bar{\pi})$.

Proposition 1. Consider the monopoly case and suppose that desity f is such that the profit function $\pi(p) = p \int_p^1 f(\theta) d\theta$ is strictly concave. The following matching scheme $\phi^*(\cdot)$ implements the point on the lower envelope with a profit $\pi \in (0, \bar{\pi})$:

$$\phi^*(\theta) = \begin{cases} 1 & \text{if } \theta \leq \underline{\theta} \\ \dfrac{\pi}{\theta^2 f(\theta)} & \text{if } \theta \in [\underline{\theta}, \bar{\theta}) \\ 1 & \text{if } \theta \geq \bar{\theta} \end{cases}$$

where $\bar{\theta}$ is the larger root of $\pi(p) = \pi$ (which has two roots), and $\underline{\theta}$ is the unique root of $\theta^2 f(\theta) = \pi$.

Proposition 1 is proved in Appendix A. For the case of uniform distribution, *SUR* is plotted in panel (a) of Figure 2; the matching ϕ^* which solves the problem for various values of π are depicted in panels (b)-(d). Paired with the consumer-optimal segmentation of Bergemann, Brooks, and Morris (2015), the welfare outcomes they implement are shown as red dots on the boundary of *SUR* in panel (a).

The matching schemes which allows us to implement the lower envelope of *SUR* has the following features: the firm has always access to consumers with a sufficiently high value ($\theta \geq \bar{\theta}$) and a sufficiently low value ($\theta \leq \underline{\theta}$), but the firm has only partial access to consumers with intermediate values. This partial access distorts the distribution over intermediate value consumers in a way that the firm is indifferent between setting any price in the intermediate region.[6]

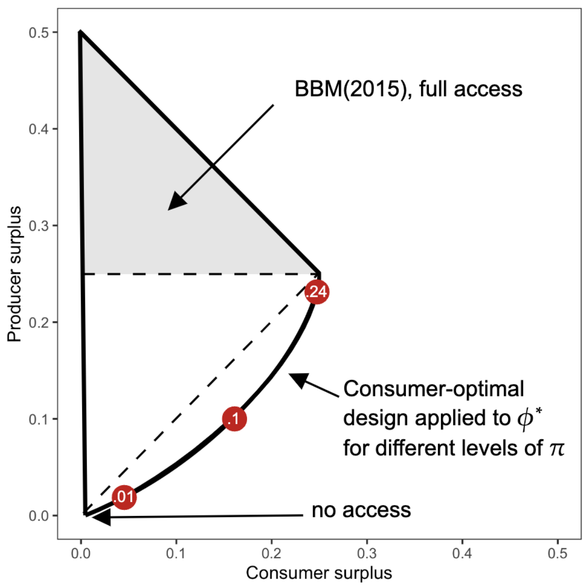
3.2 Proof of Theorem 1.

Part (i). Let all firms access only to those consumers who value their product the most (i.e., the consideration set of consumers in E_i comprises only firm i), and give firms full information about these consumers' valuations. There is then an equilibrium where each firm i sells to all consumers in E_i at a price equals to their respective valuations for product i. The outcome of this equilibrium is the producer optimal point.

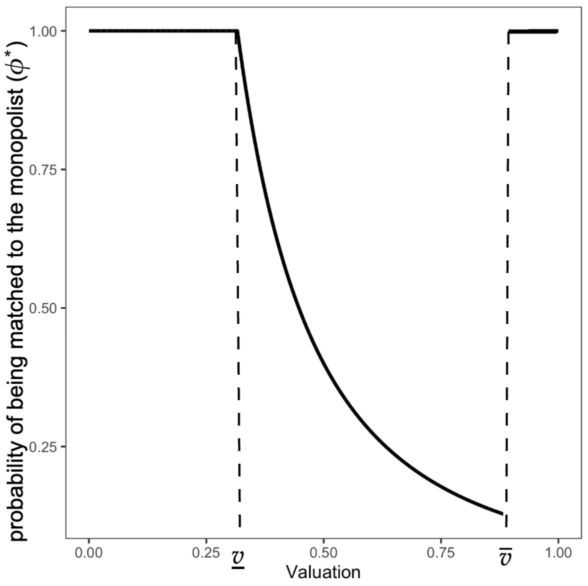
Part (ii). We characterize the consumer-optimal outcome for an arbitrary matching design in two steps. In the first step, for a given matching design, we define the *new valuations* of each consumer type $\boldsymbol{\theta}$ by setting to zero the valuation for product i whenever firm i is not in the consideration set for consumer type $\boldsymbol{\theta}$. This defines a modified economy where consumers

[6] If the density f is such that $\pi(p)$ is not strictly concave, then the set of prices that the monopoly is indifferent to charge under the ϕ- consumer-optimal design may not be an interval and could be quite complex.

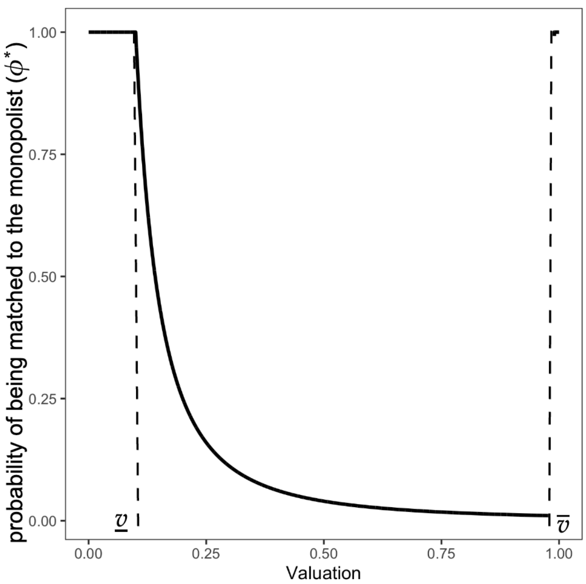
Figure 2: Illustration of *SUR* and matching schemes which implement the lower envelope



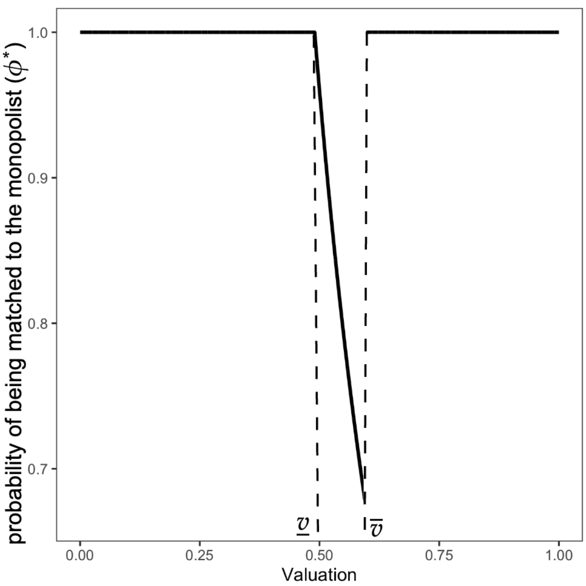
(a) *SUR* for monopoly



(b) matching scheme ϕ^* with $\pi = 0.1$



(c) matching scheme ϕ^* with $\pi = 0.01$



(d) matching scheme ϕ^* with $\pi = 0.24$

have the *new valuations* for products and consumers' consideration sets are unrestricted. In the second step we apply Theorem 2^C in the online appendix of EGKL, which characterizes the consumer-optimal information design when all consumers have all firms in their consideration sets, to the modified economy. This gives the consumer-optimal outcome of this modified economy. Lemma 1 below shows that this corresponds to the consumer-optimal outcome for the initial economy with restricted consideration sets. We now develop these arguments formally.

Step 1: Modify the distribution of valuations. A matching scheme $\phi : \Theta \to \Delta(2^{\mathcal{N}})$ maps consumer types into a probability distribution over consideration sets. For an initial consumer type $\boldsymbol{\theta}$ let $S \in 2^{\mathcal{N}}$ be her realized consideration set. We map this consumer type and consideration set pair, $(\boldsymbol{\theta}, S)$, into a new consumer type and the unrestricted consideration set pair $(\boldsymbol{\theta}^S, \mathcal{N})$, where $\boldsymbol{\theta}^S = (\theta_i^S)_{i \in \mathcal{N}}$ is such that

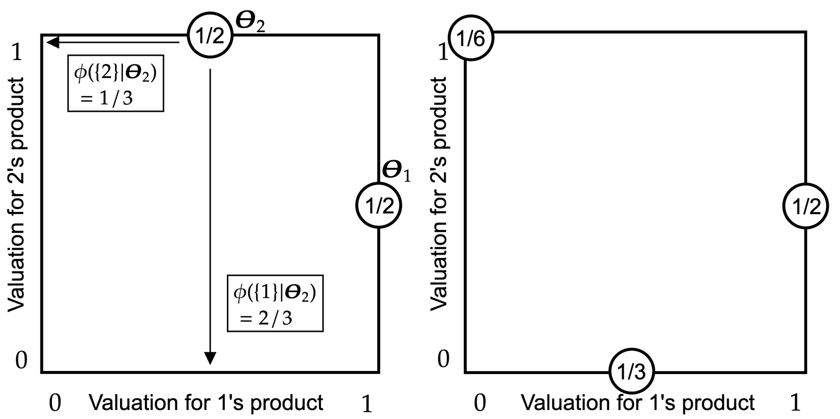
$$\theta_i^S := \begin{cases} \theta_i & \text{if } i \in S \\ 0 & \text{otherwise.} \end{cases}$$

Given the initial distribution of types, a matching scheme, and doing this mapping for all consumer type-matching scheme realization pairs induces a new ex-ante distribution μ_ϕ over consumer types along with an matching scheme that gives all consumer types unrestricted matching. We illustrate the construction in a simple example.

Example: Illustration of Step 2. There are two firms $\mathcal{N} = \{1, 2\}$, and two, equally likely, consumer's types. Consumer type $\boldsymbol{\theta}_1 = (1, 0.5)$ values product 1 the most and consumer type $\boldsymbol{\theta}_2 = (0.5, 1)$ values product 2 the most. Consider the following matching scheme:

- The consideration set of $\boldsymbol{\theta}_1$ is unrestricted, i.e., $\phi(\{1, 2\}|\boldsymbol{\theta}_1) = 1$
- The consideration set of $\boldsymbol{\theta}_2$ is firm 1 with probably 2/3 and firm 2 with probability 1/3, i.e., $\phi(\{1\}|\boldsymbol{\theta}_2) = 2/3$, $\phi(\{2\}|\boldsymbol{\theta}_2) = 1/3$.

Figure 3: Illustration of Step 1



(a) Original distribution of types (b) Modified distribution of types

The original distribution of consumer's types is given by the two-point distribution in panel (a) of Figure 3. The corresponding modified distribution of valuations in which all firms are matched to all consumers is shown in panel (b). Whenever consumer type $\boldsymbol{\theta}_1 = (1, 0.5)$

is realized all firms are already matched to the consumer so that point remains unchanged under the modification. However, when consumer type $\boldsymbol{\theta}_2 = (0.5, 1)$ is realized, initially, with probability $2/3$, only firm 1 is matched to the consumer and with probability $1/3$ only firm 2 is matched to the consumer. We modify the setting in this case by giving both firms full access to the consumer, but changing the consumer's valuations so that the consumer has a $2/3 \cdot 1/2 = 1/3$ chance of having valuations $(0.5, 0)$ for products 1 and 2 respectively, and a $1/3 \cdot 1/2 = 1/6$ chance of having valuations $(0, 1)$.

Step 2: Apply the consumer-optimal structure to the modified distribution.

For a given realization of type $\boldsymbol{\theta} \in \Theta$ and consideration set $S \in 2^{\mathcal{N}}$, we treat the consumer's type as if it were $\boldsymbol{\theta}^S$. We then assign messages to each firm as if the underlying type were $\boldsymbol{\theta}^S$. In particular, we choose

$$\pi(\cdot|\boldsymbol{\theta}^S) \in \Delta\Big([0,1]^n\Big) \quad \text{for each } \boldsymbol{\theta}^S \in \Theta$$

as the consumer-optimal information structure of EGKL as applied to the modified distribution μ_ϕ. Finally, define $\psi^\star_\phi$ as follows:

$$\psi^\star_\phi(M \times \{S\}|\boldsymbol{\theta}) := \pi(M|\boldsymbol{\theta}^S)\phi(S|\boldsymbol{\theta}) \quad \text{for all } M \in \mathcal{B}([0,1]^n), S \in 2^{\mathcal{N}}$$

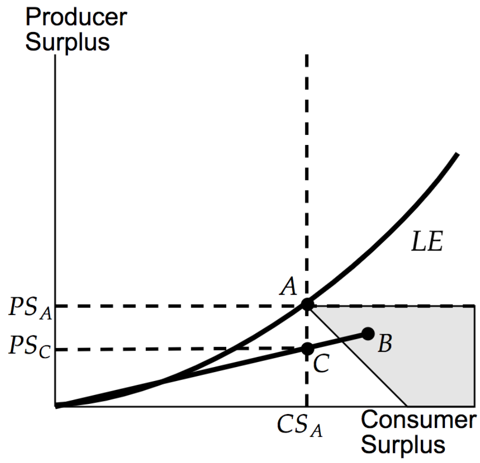
which specifies, for a given realization of type $\boldsymbol{\theta}$, a joint distribution over messages and consideration sets. By construction, $\psi^\star_\phi \in \boldsymbol{\Psi}_\phi$. The next lemma tells us that $\psi^\star$ is indeed consumer-optimal among this class of designs; the proof is in the Appendix.

Lemma 1. The design $\psi^\star_\phi$ implements an equilibrium which obtains the highest consumer surplus and the lowest producer surplus across all equilibria that can be implemented by some design in $\boldsymbol{\Psi}_\phi$, i.e., $\psi^\star_\phi$ implements the consumer-optimal outcome among $\boldsymbol{\Psi}_\phi$. Furthermore, this outcome is efficient given the matching constraints ϕ.

Fix a point $A := (CS_A, PS_A)$ on the lower envelope, and let the associated design implementing A be ψ_A. Further let ϕ_A be the matching scheme associated with ψ_A. We claim that $\psi^\star_{\phi_A}$ can also implement point A.

Suppose, towards a contradiction, that it did not. Denote by $B = (CS_B, PS_B)$ the consumer-optimal outcome that can be implemented by $\psi^\star_{\phi_A}$. From Lemma 1 we know that $CS_B \geq CS_A$ and $PS_B \leq PS_A$. In fact, it must be the case that $CS_B > CS_A$ because, if $CS_B = CS_A$ then either $PS_A = PS_B$, which contradicts that $\psi^\star_{\phi_A}$ cannot implement A or $PS_B < PS_A$, which contradicts that A is in the lower envelope of SUR. The relation between point B and point A is illustrated in Figure 4 below, where the grey area indicates the area where point B can be located. Note that the diagonal line in the picture are all the points that produce the same total surplus as point A; by Lemma 1 point B must be efficient given ϕ and hence it produces weakly higher total surplus.

Figure 4: Illustration of the proof of Proposition 4



Note next that in the producer-consumer surplus space, we can implement any convex combination of point B and the no-trade point NT by using designs which are convex combinations of $\psi^{\star}_{\phi_A}$ and the no-trade design (the consideration set of each consumer's type is empty). Since $CS_B > CS_A$ and $PS_B \leq PS_A$ there exists a convex combination of points B and NT, which we denote by $C = (CS_C, PS_C)$, such that $CS_C = CS_A$ and $PS_C < PS_A$ (see Figure 4 for graphical illustration). But this contradicts our assumption that point A belongs to the lower envelope of SUR.

Part (iii). Define the map $\psi \mapsto CS(\psi) \in \mathbb{R}_{\geq 0}$ as the highest consumer surplus achieved in any equilibrium implemented by the design ψ. Recall that, by definition, there exists a point in the lower envelope which delivers the maximum amount of consumer surplus across any design. In part (ii) we showed that every outcome in the lower envelope can be implemented by the design $\psi^{\star}_{\phi}$ for some ϕ which implies

$$\max_{\psi \in \Psi} CS(\psi) = \max_{\psi \in \{\psi^{\star}_{\phi}\}_{\phi \in \Phi}} CS(\psi)$$

It remains to show that the consumer-optimal design associated with the full matching scheme implements maximum consumer surplus across $\{\psi^{\star}_{\phi}\}_{\phi \in \Phi}$. We say that the matching scheme ϕ is efficient if for all $i \in \mathcal{N}$ and all $\boldsymbol{\theta} \in E_i$,

$$\sum_{\substack{S \in 2^{\mathcal{N}}: \\ i \in S}} \phi(S|\boldsymbol{\theta}) = 1$$

i.e., the consideration set of each consumer's type includes her favourite firm with probability one. Denote the set of efficient matching schemes with $\boldsymbol{\Phi}^E \subset \boldsymbol{\Phi}$. The following lemma shows that the solution to the consumer surplus maximization problem lies within $\boldsymbol{\Phi}^E$.

Lemma 2. For each $\phi \in \Phi \setminus \Phi^E$, there exists $\phi' \in \Phi^E$ such that $\psi^\star_{\phi'}$ implements an equilibrium outcome with strictly higher consumer surplus than $\psi^\star_\phi$. That is:

$$\max_{\psi \in \{\psi^\star_\phi\}_{\phi \in \Phi^E}} CS(\psi) > \max_{\psi \in \{\psi^\star_\phi\}_{\phi \in \Phi \setminus \Phi^E}} CS(\psi).$$

Lemma 2 is proved in Appendix A. We now show that among efficient matching schemes, the full matching scheme implements the maximum consumer surplus. To see this, observe that total surplus is the same across all efficient matching schemes since for all $\phi \in \Phi^E$,

$$TS^\phi = \int_{\boldsymbol{\theta} \in \Theta} \sum_{S \in 2^{\mathcal{N}}} \max_{j \in S} \theta_j \phi(S|\boldsymbol{\theta}) f(\boldsymbol{\theta}) d\boldsymbol{\theta} = \int_{\boldsymbol{\theta} \in \Theta} \max_{j \in \mathcal{N}} \theta_j f(\boldsymbol{\theta}) d\boldsymbol{\theta}.$$

On the other hand, from Theorem 2^C of EGKL and Lemma 1, the expected profits of firm i under the design $\psi^\star_\phi$ is

$$\underline{\Pi}^\phi = \max_{p \in [0,1]} p \cdot \int_{\boldsymbol{\theta} \in E_i} \sum_{\substack{S \in 2^{\mathcal{N}}: \\ i \in S}} \mathbf{1}(\theta_i - p \geq \max_{j \in S \setminus \{i\}} \theta_j) \phi(S|\boldsymbol{\theta}) f(\boldsymbol{\theta}) d\boldsymbol{\theta}.$$

Observe that pointwise (fixing $\boldsymbol{\theta} \in E_i$), we have that

$$\sum_{\substack{S \in 2^{\mathcal{N}}: \\ i \in S}} \mathbf{1}(\theta_i - p \geq \max_{j \in S \setminus \{i\}} \theta_j) \phi(S|\boldsymbol{\theta}) f(\boldsymbol{\theta}) \geq \mathbf{1}(\theta_i - p \geq \max_{j \in \mathcal{N} \setminus \{i\}} \theta_j) f(\boldsymbol{\theta}),$$

since restricting the consumer's consideration set decreases competition, i.e., the max operator is increasing in the set order, and efficient matching scheme ϕ has firm i in type $\boldsymbol{\theta}$'s consideration set with full probability. Hence, the design $\psi^\star_\phi$ corresponding to the full matching scheme minimizes firm i's profits and thus total producer surplus. Putting everything together, we have

$$CS^\star := \max_{\psi \in \Psi} CS(\psi) = \max_{\psi \in \{\psi^\star_\phi\}_{\phi \in \Phi}} CS(\psi) = \max_{\psi \in \{\psi^\star_\phi\}_{\phi \in \Phi^E}} CS(\psi) = CS(\psi^\star_{\phi^F})$$

where we use $\phi^F \in \Phi^E$ to denote the full matching scheme.

Part (iv). Part (i) of Theorem 1 shows that the producer optimal point $(0, TS)$ belongs to the set SUR, Part (iii) shows that the consumer optimal point is in the efficient frontier and part of the lower envelope of SUR, Part (ii) characterizes the lower-envelope. Note that SUR is convex: A $(\alpha, 1-\alpha)$ convex combination of two implementable welfare outcomes can be implemented by using the corresponding designs with respective probabilities α and $1-\alpha$. Hence, SUR is the convex hull generated by the producer optimal point and the lower envelope (LE).

3.3 Remarks. We conclude this section with two further remarks. First, every welfare outcome in SUR can be implemented through public signals. This is because we saw that every design implementing the producer optimal point and the surplus points in the lower envelope can be achieved through public signals. Second, although we considered general matching schemes, focusing our attention on matching schemes in which the consideration set of each consumer is at most two firms is in fact sufficient to implement SUR. This follows from the nature of price competition, in which the consumer's second favourite firm among her consideration set poses a necessary and sufficient constraint on her favourite firm's pricing strategy.

4 Minimum Size of Consideration Sets

We now consider what surplus points within SUR are eliminated by the constraint that all consumers must have access to at least $K \leq N$ firms. This constraint may reflect a regulation imposed on an intermediary that influences matching between firms and consumers. Alternatively, it may represent consumers search choices. For example, K could be the number of products shown on the first page of a search query in an online marketplace.

We model such settings by defining

$$\boldsymbol{\Psi}_{\geq K} := \left\{ \psi \in \boldsymbol{\Psi} : \text{for almost all } \boldsymbol{\theta} \in \Theta, S \in 2^{\mathcal{N}}, \psi([0,1]^n \times \{S\} | \boldsymbol{\theta}) > 0 \implies |S| \geq K \right\}$$

as the set of all designs in which every consumer type is shown at least K offers. We study equilibrium welfare outcomes under this restriction, which we define as follows.

Definition 3. The K-feasible surplus set $SUR_K \subset \mathbb{R}^2_{\geq 0}$ are the pairs of producer surplus (PS) and consumer surplus (CS) that can be implemented as an equilibrium outcome of some design $\psi \in \boldsymbol{\Psi}_{\geq K}$.

To make progress on this problem, we assume that consumers' valuation for each firms' good is drawn independently from a full-support continuous and density $g : [0, 1] \to [\underline{g}, \overline{g}]$ for some $\underline{g}, \overline{g} > 0$. This is a special case of the setting introduced in Section 2. Our assumption that $f(\boldsymbol{\theta}) = \prod_{i=1}^{n} g(\theta_i)$ for all $\boldsymbol{\theta} \in \Theta$ imposes (i) symmetry on the relative attractiveness of each firms' product; and (ii) independence in valuations across different firms' products.[7] Let $l : \mathcal{B}(\mathbb{R}^2) \to [0, +\infty)$ denote the Lebesgue measure.

Proposition 2. For any fixed K, when the number of firms is sufficiently large almost every outcome in SUR is achievable, i.e., For any $\epsilon > 0$ and K, there exists $\bar{n}_{\epsilon,K}$ such that for all

[7] As is clear from the analysis, we could weaken this slightly to having some correlation or having firm-specific distributions, i.e., value for firm i drawn from g_i independently.

$n \geq n_{\epsilon,K}$,

$$l\Big(SUR \setminus SUR_K\Big) \leq \epsilon.$$

Moreover, there exists n_K such that every point in the efficient frontier of SUR can be implemented exactly for all $n \geq n_K$.

To gain intuition for the first part of the result consider the valuation a consumer places on her K least preferred products. Her most preferred product of these K has a value distributed according to the $n-K$-th order statistic of the distribution values are drawn from, and hence the value of this product gets small as n gets large. This means that for each consumer there are K products that they value very little. Matching a consumer to one of these products instead of a product she values is then almost equivalent to removing access to the valued product. Thus, the restriction ensuring the consumer can access at least K products is superfluous.

The second part of Proposition 2 is more subtle. For example, to obtain exactly the producer optimal outcome, the consumer must buy her favorite product and receive no surplus from doing so. This implies that she would be willing to pay a strictly positive price for any other product that she values positively, and for finite n she will have access to such a product. What then stops the producer of this product from selling to the consumer at such a price?

The key here is the information structure. Suppose that producers know only the value the consumer places on her most preferred product, and not whether their product is the one the consumers most prefers or one of the other $K-1$ products. Moreover, suppose the information structure is designed so that all firms consider themselves equally likely to be the producer of the most preferred product. These firms can then be incentivized to all set a price equal to the consumer's value for her most preferred product (such that the consumer would get negative consumer surplus from buying any product other than her most preferred product). If the firms all set this price, then one of them will get lucky and sell to the consumer at a price equal to her valuation. To minimize the incentives of the firm to deviate and set a lower price, we can match the consumer to her $K-1$ least favourite products, as well as her favorite product in effect polarizing the consumer's preferences. Then, for n sufficiently large, a $1/K$ chance of selling to the consumer at her valuation for her most preferred product is better than selling to the consumer for sure at a price equal to her valuation for any other product she has access to, and this ensures that no firm has a profitable deviation.

5 Concluding remarks

We have considered a marketplace in which sellers offer differentiated products and compete in prices for consumers. Different designs of the marketplace affect the buyer's and the sellers' matching opportunities and the information that sellers have about the buyer's valuation. We have characterised the combination of producer and consumer surplus that can be implemented across all possible designs.

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Appendix to 'Matching and Information Design in Marketplaces'

A Omitted proofs in Section 3

A.1 Proof of Lemma 1. We provide the proof of Lemma 1, which is part of the proof of Theorem 1.

Proof of Lemma 1. We apply the consumer-optimal information structure of EGKL (Theorem 2^C, EGKL (2021)[8]) to the modified measure μ_ϕ and, therefore, the equilibrium outcome is efficient given μ_ϕ: for each pair $(\boldsymbol{\theta}, S)$, the new type $\boldsymbol{\theta}^S$ purchases from her favourite firm among the set of firms S, i.e., $\boldsymbol{\theta}^S$ buys from firm $j \in \operatorname{argmax} \theta_i^S = \operatorname{argmax}_{i \in S} \theta_i$. Hence, the design implements an equilibrium that extracts all gains from trade given ϕ. These are:

$$TS^\phi := \int_{\boldsymbol{\theta} \in \Theta} \big(\max_{i=1,2,\ldots,n} \theta_i \big) \mu_\phi(d\boldsymbol{\theta})$$

where we integrate against the modified measure μ_ϕ (which captures the fact that under the realization $(\boldsymbol{\theta}, S)$, the consumer only has positive valuation for firms in the set S).

Further, fixing the matching scheme ϕ, notice that the minimum profits firm i can make across any information structure is

$$\underline{\Pi}_i^\phi := \sup_{p \in [0,1]} p \cdot \int_{\boldsymbol{\theta} \in E_i} \mathbf{1}(\theta_i - p \geq \max_{j \neq i} \theta_j) \mu_\phi(d\boldsymbol{\theta})$$

which is the profit that firm i makes when all other firms charge a price of zero and firm i chooses an optimal uniform price against the residual demand curve. Note here that this continues to integrate over the set E_i, but under the modified measure μ_ϕ. This is because firm i will not make any sale to consumers outside E_i when all other firms are charging a price of zero.

But since (i) $\psi_\phi^\star$ was constructed such that firm i's profits are held down to $\underline{\Pi}_i^\phi$; and (ii) the allocation is efficient given ϕ, this must imply that consumer surplus

$$CS^\phi = TS^\phi - \sum_{i=1}^{n} \underline{\Pi}_i^\phi$$

is optimal. Further note that point (i) implies the consumer-optimal outcome leads to the lowest possible producer surplus across all equilibria that can be sustained by some design in $\boldsymbol{\Psi}_\phi$. □

[8] For easy reference we provide the proof of Theorem 2^C of EGKL in the supplementary material.

A.2 Proof of Lemma 2. We provide the proof of Lemma 2, which is part of the proof of Theorem 1.

Proof of Lemma 2. Since ϕ is inefficient, there exists some firm i and some positive measure of types in E_i who, with strictly positive probability, do not have firm i in their consideration set under ϕ. Since the number of firms are finite, this, in turn, implies that there exists some firm $j \neq i$ such that there is a positive measure of types within E_i which with with strictly positive probability, (i) do not have i in their consideration set; (ii) have j in their consideration set; and (iii) prefer j to all other firms in their consideration set.

Denote the type-consideration set pairs which fulfil this condition with

$$T_{ij} := \Big\{(\boldsymbol{\theta}, S) \in \Theta \times 2^{\mathcal{N}} : \boldsymbol{\theta} \in E_i, j \in S, \theta_j > \max_{k \in S \setminus \{j\}} \theta_k\Big\},$$

observing that, by the argument above,

$$\int_{\boldsymbol{\theta} \in [0,1]^n} \sum_{S:(\boldsymbol{\theta},S) \in T_{ij}} \phi(S|\boldsymbol{\theta}) f(\boldsymbol{\theta}) d\boldsymbol{\theta} > 0.$$

We will proceed by showing that suitably modifying the access scheme to give firm i access to types $\boldsymbol{\theta} : (\boldsymbol{\theta}, S) \in T_{ij}$ strictly improves consumer welfare.

Denote

$$F_i := \Big\{(\boldsymbol{\theta}, S) \in \Theta \times 2^{\mathcal{N}} : i \in S, \theta_i > \max_{k \in S \setminus \{i\}} \theta_k\Big\}$$

as the type-consideration pairs where the consumer type strictly prefers product i to other products in her consideration set. Note that $T_{ij} \cap F_i = \emptyset$.

In the consumer-optimal outcome implemented by $\psi^\star_\phi$ firm i's profit is

$$\pi_i(\psi^\star_\phi) := \max_{p \in [0,1]} p \int \sum_{S:(\boldsymbol{\theta},S) \in F_i} \mathbf{1}(\theta_i - p \geq \max_{k \in S \setminus \{i\}} \theta_k) \phi(S|\boldsymbol{\theta}) f(\boldsymbol{\theta}) d\boldsymbol{\theta}.$$

Let us now modify the inefficient access scheme ϕ as follows: on the event $S|\boldsymbol{\theta}$ where $(\boldsymbol{\theta}, S) \in T_{ij}$, implement instead the consideration set $S \cup \{i\}$ and denote the resultant access scheme with ϕ', i.e. on each $\boldsymbol{\theta} : (\boldsymbol{\theta}, S) \in T_{ij}$, we have

$$\phi'(S \cup \{i\}|\boldsymbol{\theta}) = \phi(S|\boldsymbol{\theta}).$$

A few observations follow. First, notice that on the realizations of $(\boldsymbol{\theta}, S) \in T_{i,j}$ under the design $\psi^\star_\phi$ those consumers bought from j. However, under the design $\psi^\star_{\phi'}$ they now buy from i. This implies j's profits must weakly decrease i.e., $\pi_j(\psi^\star_{\phi'}) \leq \pi_j(\psi^\star_\phi)$. Second, observe that under the design $\psi^\star_{\phi'}$, i's profits are now

$$\pi_i(\psi^\star_{\phi'}) := \max_{p\in[0,1]} p \int \sum_{S:(\boldsymbol{\theta},S)\in\{F_i\cup T_{ij}\}} \mathbf{1}(\theta_i - p \geq \max_{k\in S\setminus\{i\}} \theta_k)\phi(S|\boldsymbol{\theta})f(\boldsymbol{\theta})d\boldsymbol{\theta}$$
$$\leq \pi_i(\psi^\star_{\phi}) + \max_{p\in[0,1]} p \int \sum_{S:(\boldsymbol{\theta},S)\in T_{ij}} \mathbf{1}(\theta_i - p \geq \theta_j)\phi(S|\boldsymbol{\theta})f(\boldsymbol{\theta})d\boldsymbol{\theta},$$

where the inequality comes from applying the max operator to each term in the summand separately. We can then bind the improvement to i's profits by the change in total surplus as follows:

$$\pi_i(\psi^\star_{\phi'}) - \pi_i(\psi^\star_{\phi}) \leq \max_{p\in[0,1]} p \int \sum_{S:(\boldsymbol{\theta},S)\in T_{ij}} \mathbf{1}(\theta_i - p \geq \max_{k\in S\setminus\{i\}} \theta_k)\phi(S|\boldsymbol{\theta})f(\boldsymbol{\theta})d\boldsymbol{\theta}$$
$$= \int \sum_{S:(\boldsymbol{\theta},S)\in T_{ij}} p^*\mathbf{1}(\theta_i - \theta_j \geq p^*)\phi(S|\boldsymbol{\theta})f(\boldsymbol{\theta})d\boldsymbol{\theta}$$
$$< \int \sum_{S:(\boldsymbol{\theta},S)\in T_{ij}} (\theta_i - \theta_j)\phi(S|\boldsymbol{\theta})f(\boldsymbol{\theta})d\boldsymbol{\theta}$$
$$= TS(\psi^\star_{\phi'}) - TS(\psi^\star_{\phi}),$$

where we use p^* to denote the solution of the maximization problem in the first equality and the second inequality follows from noting that $p^*\mathbf{1}(\theta_i - \theta_j \geq p^*) \leq \theta_i - \theta_j$. This inequality is strict for a strictly positive measure subset of T_{ij}.[9] The last equality follows by observing that the increase in gains from trade obtained with the new design corresponds to the increase in consumption value obtained by the consumers who now purchase from i instead of j.

Denote $PS(\psi^\star_{\phi})$ as the producer surplus implemented by $\psi^\star_{\phi}$. We have:

$$PS(\psi^\star_{\phi'}) - PS(\psi^\star_{\phi}) = \sum_{k\in\mathcal{N}} \left(\pi_k(\psi^\star_{\phi'}) - \pi_k(\psi^\star_{\phi})\right)$$
$$= \left(\pi_i(\psi^\star_{\phi'}) - \pi_i(\psi^\star_{\phi})\right) + \left(\pi_j(\psi^\star_{\phi'}) - \pi_j(\psi^\star_{\phi})\right)$$
$$< TS(\psi^\star_{\phi'}) - TS(\psi^\star_{\phi}),$$

where the inequality follows from the argument above that j's profits decrease and i's profits increase by less than the change in total surplus.

[9]Conditional on set T_{ij}, if $\theta_i - \theta_j < p^*$ almost surely, then we have $p^*\mathbf{1}(\theta_i - \theta_j \geq p^*) = 0 < \theta_i - \theta_j$ almost surely because $\theta_i - \theta_j > 0$. If $\theta_i - \theta_j \geq p*$ with strictly positive probability, then since μ is full support over Θ, it must be that $\theta_i - \theta_j > p^*$ with strictly positive probability, hence $p^*\mathbf{1}(\theta_i - \theta_j \geq p^*) < \theta_i - \theta_j$ with strictly positive probability.

Now denoting $CS(\psi^\star_\phi)$ as total consumer surplus under $\psi^\star_\phi$, we have

$$\begin{aligned} CS(\psi^\star_{\phi'}) - CS(\psi^\star_\phi) &= \Big(TS(\psi^\star_{\phi'}) - PS(\psi^\star_{\phi'})\Big) - \Big(TS(\psi^\star_\phi) - PS(\psi^\star_\phi)\Big) \\ &= \Big(TS(\psi^\star_{\phi'}) - TS(\psi^\star_\phi)\Big) - \Big(PS(\psi^\star_{\phi'}) - PS(\psi^\star_\phi)\Big) > 0. \end{aligned}$$

Finally, note that we can repeat this procedure a finite number of times until the final matching scheme is efficient. □

A.3 Proof of Proposition 1. We now provide a proof of Proposition 1, which characterises the matching schemes used to implement points on the lower envelope of SUR in the monopoly case.

Proof of Proposition 1. In order implement a point on the lower envelope with profit π, the corresponding matching scheme solves the following problem:

$$\sup_{\phi:[0,1]\to[0,1]} \int_0^1 \theta\phi(\theta)f(\theta)d\theta \quad \text{s.t.} \quad \max_p p \cdot \int_p^1 \phi(\theta)f(\theta)d\theta \le \pi.$$

In words, we wish to hold the monopolist's profits from uniform prices down to π while maximizing the amount of available total surplus. We now show that the solution to this program is, indeed, given by Proposition 1.

Step 1: Showing that ϕ^*, $\underline{\theta}$, and $\overline{\theta}$ are well-defined. Since $\pi(p)$ is strictly concave, $\pi(p) = \pi$ has exactly two roots. Hence, $\bar{\theta}$ is well defined and $\bar{\theta} \in (p^*, 1)$. Recall that p^* is the optimal uniform price $p^* = \text{argmax}_p\, \pi(p)$.

To show $\underline{\theta}$ is well defined. Note that:

$$\frac{dp^2 f(p)}{dp} = p\Big(2f(p) + pf'(p)\Big) = p\Big(-\pi''(p)\Big) > 0,$$

hence, $p^2 f(p)$ is strictly increasing. Since $p^2 f(p)\big|_{p=0} = 0$, we now show $p^{*2} f(p^*) > \pi$ to conclude that $\underline{\theta}$ is well defined, which is the unique root of $p^2 f(p) = \pi$ and $\underline{\theta} \in (0, p^*)$. Note:

$$\pi'(p^*) = 0 = \int_{p^*}^1 f(\theta)d\theta - p^* f(p^*).$$

Hence,

$$p^{*2} f(p^*) = p^* \int_{p^*}^1 f(\theta)d\theta = \pi(p^*) > \pi.$$

To see $\phi^*(\cdot)$ is well defined, note that by definition of $\underline{\theta}$ and our observation that $p^2 f(p)$ is strictly increasing, $p^2 f(p) \geq \pi$ for $p \geq \underline{\theta}$, hence,

$$\phi^*(\theta) = \frac{\pi}{\theta^2 f(\theta)} \leq 1, \text{ for } \theta \in [\underline{\theta}, \bar{\theta}).$$

Step 2: The solution $\phi^*(\cdot)$ satisfies the constraint. For $p \geq \bar{\theta}$,

$$p \cdot \int_p^1 \phi^*(\theta) f(\theta) d\theta = p \cdot \int_p^1 f(\theta) d\theta = \pi(p) \leq \pi(\bar{\theta}) = \pi.$$

The inequality is because profit function $\pi(p)$ decreases in the interval $[\bar{\theta}, 1]$. Note the inequality holds with equality for $p = \bar{\theta}$, hence,

$$\bar{\theta} \cdot \int_{\bar{\theta}}^1 \phi^*(\theta) f(\theta) d\theta = \pi. \tag{1}$$

For $p \in [\underline{\theta}, \bar{\theta})$,

$$p \cdot \int_p^1 \phi^*(\theta) f(\theta) d\theta = p \cdot \Big[\int_p^{\bar{\theta}} \phi^*(\theta) f(\theta) d\theta + \int_{\bar{\theta}}^1 \phi^*(\theta) f(\theta) d\theta \Big] = p \cdot \Big[\int_p^{\bar{\theta}} \frac{\pi}{\theta^2} d\theta + \frac{\pi}{\bar{\theta}} \Big] = \pi. \tag{2}$$

The second equality is from the fact that $\phi^*(\theta) = \dfrac{\pi}{\theta^2 f(\theta)}$ for $\theta \in [\underline{\theta}, \bar{\theta})$ and equation (1).

The case $p \in [0, \underline{\theta})$ can be shown analogously to the case $p \in [\underline{\theta}, \bar{\theta})$, by replacing the second equality in equation (2) with a weak inequality ($\leq$) since for $\theta \in [p, \underline{\theta})$, $\phi^*(\theta) = 1$ and $f(\theta) < \pi/\theta^2$. Hence, the constraint is satisfied pointwise.

Step 3: Optimaility of $\phi^*(\cdot)$. From integration by parts, the objective function can be rewritten as follows:

$$\int_0^1 \theta \phi(\theta) f(\theta) d\theta = \int_0^1 \int_p^1 \phi(\theta) f(\theta) d\theta dp.$$

Solution $\phi^*(\cdot)$ is optimal since it pointwise maximizes function $\int_p^1 \phi(\theta) f(\theta) dv$ subject to the constraint: (i) for $p \in [\bar{\theta}, 1]$, $\phi^*(\theta) = 1$ for each $\theta \in [p, 1]$, hence $\int_p^1 \phi(\theta) f(\theta) d\theta$ is maximized; (ii) for $p \in [\underline{\theta}, \bar{\theta})$, $p \cdot \int_p^1 \phi^*(\theta) f(\theta) d\theta = \pi$ (by equation (2)), hence the constraint binds exactly; and (iii) for $p \in [0, \underline{\theta})$,

$$\int_p^1 \phi(\theta) f(\theta) dv = \int_p^{\underline{\theta}} \phi(\theta) f(\theta) d\theta + \int_{\underline{\theta}}^1 \phi(\theta) f(\theta) d\theta \leq \int_p^{\underline{\theta}} f(\theta) d\theta + \frac{\pi}{\underline{\theta}}.$$

The last inequality is by $\phi(\theta) \leq 1$ and $\underline{\theta} \cdot \int_{\underline{\theta}}^{1} \phi(\theta) f(\theta) d\theta \leq \pi$ (from the constraint). Note that $\phi^*(\cdot)$ exactly achieves this upper bound. □

B Omitted Proofs from Section 4

Proof of Proposition 2. For a given realization of consumer type $\boldsymbol{\theta} \in \Theta$, order $\theta_{(1)} > \theta_{(2)} > \ldots > \theta_{(n)}$ where we let $\boldsymbol{\theta}_{(j)}$ be the j−th highest element. Further define $(j)_{\boldsymbol{\theta}}$ as the j-th favourite firm of type $\boldsymbol{\theta}$. Note that because the distribution is atomless, there are no ties almost surely.

Lemma 3 shows that for sufficiently large n the producer optimal outcome is exactly $PO = (0, TS)$

Lemma 3. For every given K, there exists $\bar{n} > 0$ such that if the number of firms is larger than $\bar{n}$, the producer-optimal point $PO = (0, TS)$ is implementable. The producer-optimal point can be implemented with the following design:

(i) The consideration set of each consumer type $\boldsymbol{\theta}$ comprises its favourite firm and the $K-1$ least favourite firms;

(ii) A public message is sent for every consumer type $\boldsymbol{\theta}$. The message reveals the consumer's valuation for her most preferred product, without revealing the identity of her most preferred firm.

Proof of Lemma 3. We wish to show that facing the modified distribution of valuations under the matching scheme specified in part (i) of Lemma 3, each firm i, upon receipt of the public message $m \in [0, 1]$, prefers to obey and charge $p = m$, given that all other firms charge m.

We begin with several observations. First, if the firm i charges m, its expected revenue is m/n (by symmetry, each firm is equally likely to be the consumer's favourite). Second, if the firm deviates to some price $p \in (0, m)$, notice that by the matching scheme we specified, the only event on which the firm could potentially business-steal is when it is among the $K - 1$ least favourite firms for the consumer. We can write firm i's payoff from deviating to the price $p \in (0, m)$ as follows:

$$REV(p) = \frac{p}{n} + p \cdot \sum_{j=1}^{K-1} \mathbb{P}(\theta_i \geq p \text{ and } \theta_i \text{ is } j\text{-lowest} \mid \theta_{(n)} = m)$$

$$= \frac{p}{n} + p \cdot \frac{\overbrace{\sum_{j=1}^{K-1} \int_p^m \binom{n-2}{j-1} \Big(G(m) - G(q)\Big)^{n-j-1} G(q)^{j-1} g(q)(n-1) g(m)\, dq}^{\text{PDF of } \theta_i = q \text{ and } \theta_i \text{ is } j\text{-th lowest draw and max is } m}}{\underbrace{n g(m) G(m)^{n-1}}_{\text{prob. highest draw is } m}}$$

$$= \frac{p}{n} + p \cdot \frac{\sum_{j=1}^{K-1} \int_p^m \binom{n-2}{j-1} \Big(G(m) - G(q)\Big)^{n-j-1} G(q)^{j-1} g(q)(n-1) dq}{n G(m)^{n-1}}$$

$$= \frac{p}{n} + p \cdot \frac{n-1}{n} \cdot \frac{\sum_{j=1}^{K-1} \int_p^m \binom{n-2}{j-1} \Big(1 - G(q)/G(m)\Big)^{n-j-1} [G(q)/G(m)]^{j-1} g(q) dq}{G(m)}$$

For firm i to obey recommendation m, we need $\frac{m}{n} \geq REV(p)$ for all $p < m$; Equivalently:

$$(m-p)G(m) \geq p \cdot (n-1) \cdot \int_p^m I(q, m, n, K) g(q) dq, \qquad \text{(Obedience constraint)}$$

where

$$I(q, m, n, K) := \sum_{j=1}^{K-1} \binom{n-2}{j-1} \Big(1 - G(q)/G(m)\Big)^{n-j-1} [G(q)/G(m)]^{j-1}.$$

Our goal will be to show the existence of a positive constant n^{PPO} such that for all $n \geq n^{PPO}$ a deviation to any price $p < m$ is unprofitable, , for any realization m. Our approach is to split potential deviations into two cases. The first case considers deviations to prices below $m/(c\sqrt{n})$ where c is a constant independent of m and n that we specify appropriately later. The second case considers deviations to prices between $m/(c\sqrt{n})$ and m.

Case 1: Deviating to $p \leq \frac{m}{c\sqrt{n}}$ is unprofitable. Split the integral in the obedience constraint into two intervals: $\left(p, \frac{m}{c\sqrt{n}}\right]$ and $\left(\frac{m}{c\sqrt{n}}, m\right]$. The integral over the first interval is at most $\bar{g}\frac{m}{c\sqrt{n}}$, since $I(q, m, n, K) \leq I(q, m, n, n) = 1$ from the Binomial theorem. For the second part, notice that:

$$\binom{n-2}{j-1} \leq \binom{K-2}{j-1} \cdot \binom{n-2}{K-2}$$

and $I(q, m, K, K) = 1$ which implies $I(q, m, n, K) \leq \binom{n-2}{K-2} \Big(1 - G(q)/G(m)\Big)^{n-K}$. Hence, we can bind the second integral as follows:

$$\begin{aligned}
\int_{m/(c\sqrt{n})}^{m} I(q,m,n,K)g(q)dq &\leq \int_{m/(c\sqrt{n})}^{m} \binom{n-2}{K-2}\Big(1-G(q)/G(m)\Big)^{n-K} g(q)dq \\
&\leq \binom{n-2}{K-2}\Big(1-G(m/(c\sqrt{n}))/G(m)\Big)^{n-K} \int_{m/(c\sqrt{n})}^{m} g(q)dq \\
&\leq \binom{n-2}{K-2}\Big(1-G(m/(c\sqrt{n}))/G(m)\Big)^{n-K} \bar{g}\cdot\Big(1-\frac{1}{c\sqrt{n}}\Big)m \\
&\leq \binom{n-2}{K-2}\left(1-\frac{\int^{m/(c\sqrt{n})} g(q)dq}{\int^{m} g(q)dq}\right)^{n-K} \bar{g}\cdot\Big(1-\frac{1}{c\sqrt{n}}\Big)m \\
&\leq n^{K-2}\Big(1-\frac{\underline{g}}{c\bar{g}\sqrt{n}}\Big)^{n-K}\bar{g}m.
\end{aligned}$$

Hence, for $p < \dfrac{m}{c\sqrt{n}}$, our obedience equation is fulfilled from the following series of inequalities:

$$\begin{aligned}
(m-p)G(m) &\geq m\Big(1-\frac{1}{c\sqrt{n}}\Big)\underline{g}m && (G(m)\geq \underline{g}m) \\
&\geq \Big[\frac{1}{c^2}+\frac{1}{c}n^{K-1}\Big(1-\frac{\underline{g}}{c\bar{g}\sqrt{n}}\Big)^{n-K}\Big]\bar{g}m^2 \\
& && \text{(By choosing } c \text{ appropriately and taking } n \text{ large; see below)} \\
&\geq \frac{m}{c\sqrt{n}}(n-1)\cdot\Big[\bar{g}\frac{m}{c\sqrt{n}}+n^{K-2}\Big(1-\frac{\underline{g}}{c\bar{g}\sqrt{n}}\Big)^{n-K}\bar{g}m\Big] \\
&\geq p\cdot(n-1)\cdot\Big(\int_{p}^{m/(c\sqrt{n})} I(q,m,n,K)g(q)dq+\int_{m/(c\sqrt{n})}^{m} I(q,m,n,K)g(q)dq\Big) \\
& && \text{(Bounds for each integral developed above)} \\
&\geq p\cdot(n-1)\int_{p}^{m} I(q,m,n,K)g(q)dq,
\end{aligned}$$

where the second inequality requires

$$1 \geq \Big[\frac{1}{c^2}+\frac{1}{c}n^{K-1}\Big(1-\frac{\underline{g}}{c\bar{g}\sqrt{n}}\Big)^{n-K}\Big]\cdot\frac{\bar{g}}{\underline{g}}+\frac{1}{c\sqrt{n}}.$$

The following fact is useful

Claim 1.

$$\lim_{n\to\infty} n^{K-1}\left(1-\frac{\underline{g}}{c\bar{g}\sqrt{n}}\right)^{n-K}=0.$$

Proof of Claim 1.

$$\left(1-\frac{\underline{g}}{c\bar{g}\sqrt{n}}\right)^{n-K}=\exp\left((n-K)\cdot\log\left(1-\frac{\underline{g}}{c\bar{g}\sqrt{n}}\right)\right)$$

now observe that we can rewrite

$$\log\left(1-\frac{\underline{g}}{c\bar{g}\sqrt{n}}\right)=\frac{1}{1+x^*}\cdot\left(\frac{-\underline{g}}{c\bar{g}\sqrt{n}}\right)\leq\frac{-\underline{g}}{c\bar{g}\sqrt{n}}\quad\text{for some } x^*\in\left(\frac{-\underline{g}}{c\bar{g}\sqrt{n}},0\right).$$

Putting things together,

$$n^{K-1}\left(1-\frac{\underline{g}}{c\bar{g}\sqrt{n}}\right)^{n-K}\leq n^{K-1}\cdot\exp\left(\frac{-\underline{g}}{c\bar{g}\sqrt{n}}\cdot(n-K)\right)$$

which converges to zero as required. □

Choose $c=(\bar{g}/3\underline{g})^{1/2}$ and note that we can find $\underline{n}_1$ such that for all $n\geq\underline{n}_1$,

$$\max\left\{\frac{1}{c}n^{K-1}\left(1-\frac{\underline{g}}{c\bar{g}\sqrt{n}}\right)^{n-K}\cdot\frac{\bar{g}}{\underline{g}},\frac{1}{c\sqrt{n}}\right\}\leq\frac{\bar{g}}{3c^2\underline{g}}$$

Hence, deviations to $p<m/(c\sqrt{n})$ are unprofitable for all $n\geq\underline{n}_1$.

Case 2: Deviating to $p>\dfrac{m}{c\sqrt{n}}$ is unprofitable. The obedience constraint can be equivalently expressed as:

$$\frac{(m-p)}{p}-\frac{n-1}{G(m)}\int_p^m I(q,m,n,K)g(q)dq\geq 0.$$

Notice that this condition is satisfied for $p=m$. Hence, it is sufficient to show that

$$\frac{\partial}{\partial p}\left[\frac{(m-p)}{p}-\frac{n-1}{G(m)}\int_p^m I(q,m,n,K)g(q)dq\right]\leq 0$$

because this would guarantee that the obedience constraint is satisfied for any $p\in(m/(c/\sqrt{n}),1]$.

We show this next:

$$\frac{\partial}{\partial p}\left[\frac{(m-p)}{p}-\frac{n-1}{G(m)}\int_p^m I(q,m,n,K)g(q)dq\right]$$

$$= -\frac{m}{p^2}+\frac{n-1}{G(m)}I(p,m,n,K)g(p) = \frac{1}{G(m)}\left[-\frac{mG(m)}{p^2}+(n-1)I(p,m,n,K)g(p)\right]$$

$$\leq \frac{1}{G(m)}\left[-\frac{m\underline{g}m}{m^2}+(n-1)\binom{n-2}{K-2}\Big(1-G(p)/G(m)\Big)^{n-K}g(p)\right]$$

$$\text{(from } I(q,m,n,K)\leq \tbinom{n-2}{K-2}\Big(1-G(q)/G(m)\Big)^{n-K}\text{)}$$

$$\leq \frac{1}{G(m)}\left[-\underline{g}+n^{K-1}\Big(1-\frac{\underline{g}}{c\bar{g}\sqrt{n}}\Big)^{n-K}\bar{g}\right]$$

From Claim 1, there exists some $\underline{n}_2$ such that for all $n \geq \underline{n}_2$

$$\frac{1}{G(m)}\left[-\underline{g}+\underbrace{n^{K-1}\Big(1-\frac{\underline{g}}{c\bar{g}\sqrt{n}}\Big)^{n-K}}_{\to 0 \text{ by Lemma 1}}\bar{g}\right]\leq 0,$$

which implies that for all $n \geq \underline{n}_2$

$$\frac{\partial}{\partial p}\left[\frac{(m-p)}{p}-\frac{n-1}{G(m)}\int_p^m I(q,m,n,K)g(q)dq\right]\leq 0$$

Combining the analyses for case 1 and case 2 we obtain that all price deviations are deterred for $n \geq n^{PPO} := \max\{\underline{n}_1, \underline{n}_2\}$ which completes the proof. □

The next Lemma 4 shows that we can obtain an arbitrarily good approximation of the no trade surplus point $NT = (0,0)$. This is done by picking the K least favourite firms for each type's consideration set so that the total gains from trade under this matching scheme converges to zero.

Lemma 4. For all $\epsilon > 0$, there exists $\bar{n}_{\epsilon,K} > 0$ such that for all $n > \bar{n}_\epsilon$, there exists $\psi^{NT} \in \mathbf{\Psi}_{\geq K}$ in which for all equilibria, both consumer and producer surplus is upper-bounded by ϵ hence this approximates the NT point $(0,0)$. ψ^{NT} takes the following form: For type $\boldsymbol{\theta}$,

(i) The consideration set of $\boldsymbol{\theta}$ comprises its K least favourite firms i.e.,

$$\phi^{NT}(S|\boldsymbol{\theta}) := \text{marg}_S\psi^{NT}(\cdot, S|\boldsymbol{\theta}) \text{ puts full probability on } \Big\{(n{-}K{+}1)_{\boldsymbol{\theta}}, (n{-}K{+}2)_{\boldsymbol{\theta}}, \dots (n)_{\boldsymbol{\theta}}\Big\}.$$

(ii) For type $\boldsymbol{\theta}$ and consideration set S, send an arbitrary public message to firms.

Proof of Lemma 4. First observe that under the matching scheme specified in part (i) of Proposition 4, the expected gains from trade are given by the $n-K+1$th highest realization, θ_{n-K+1}. As such,

$$\max\{CS, PS\} \leq \mathbb{E}[\theta_{n-K+1}] \leq \epsilon/2 + \mathbb{P}(\theta_{n-K+1} > \epsilon/2) \cdot 1 \leq \epsilon/2 + \underbrace{\binom{n}{K}(1-G(\epsilon/2))^{n-K}}_{\to 0 \text{ as } n \to +\infty.}$$

so pick n_ϵ^{NT} such that for all $n \geq n_\epsilon^{NT}$ the second term is $\leq \epsilon/2$ which completes the proof. □

Finally, Lemma 5 shows that the consumer optimal point can be implemented approximately through some design $\psi \in \mathbf{\Psi}_{\geq K}$.

Lemma 5. For all $\epsilon > 0$, there exists $\bar{n}_{\epsilon,K} > 0$ such that for all $n > \bar{n}_\epsilon$, there exists $\psi^{CO} \in \mathbf{\Psi}_{\geq K}$ which implements the equilibria with welfare outcome (CS, PS) such that $CS \geq 1-\epsilon$, and $PS \leq \epsilon$. As such, this approximates the consumer optimal point CO. ψ^{CO} takes the following form. For type $\boldsymbol{\theta}$,

(i) The consideration set of $\boldsymbol{\theta}$ comprises its K most favourite firms i.e.,

$$\phi^{NT}(S|\boldsymbol{\theta}) := \text{marg}_S \psi^{NT}(\cdot, S|\boldsymbol{\theta}) \text{ puts full probability on } \Big\{(1)_{\boldsymbol{\theta}}, (2)_{\boldsymbol{\theta}}, \dots (K)_{\boldsymbol{\theta}}\Big\}.$$

(ii) For type $\boldsymbol{\theta}$ and consideration set S, send the public message $\boldsymbol{\theta}$ to all firms i.e., give firms full information.

Proof of Lemma 5. Observe that there is a Bayes Correlated Equilibrium induced by the information structure where all firms but the consumer's favourite firm $(1)_{\boldsymbol{\theta}}$ charges a price of zero, and firm $(1)_{\boldsymbol{\theta}}$ charges the price $\theta_{(1)} - \theta_{(2)}$ and the consumer breaks ties in favour of her favourite firm.

First observe that conditioned on the highest draw $\theta_{(1)} =: m \geq \epsilon/6$, the probability that the second highest draw is greater than $\epsilon/6$ away is

$$\begin{aligned}\mathbb{P}(\theta_{(1)} - \theta_{(2)} \geq \epsilon/6 \mid \theta_{(1)} = m) &= \mathbb{P}(n-1 \text{ independent draws are} \leq m - \epsilon/6 \mid \theta_{(1)} = m) \\ &= \bigg(\frac{G(m-\epsilon/6)}{G(m)}\bigg)^{n-1} \leq \bigg(\frac{G(m) - \underline{g}\epsilon/6}{G(m)}\bigg)^{n-1} \leq \Big(1 - \underline{g}\epsilon/6\Big)^{n-1}\end{aligned}$$

which tends to zero as $n \to \infty$. Since this is true for all $m \in [\epsilon/6, 1]$, there exists $\bar{n}_{1,\epsilon}$ (which does not depend on m) such that for all $n \geq \bar{n}_{1,\epsilon}$ and all $m \in [\epsilon/6, 1]$,

$$\mathbb{P}(\theta_{(1)} - \theta_{(2)} \geq \epsilon/6 \mid \theta_{(1)} = m) \leq \epsilon/6.$$

Now applying the law of total expectation twice,

$$\begin{aligned}\mathbb{E}[\theta_{(1)}-\theta_{(2)}] &= \mathbb{E}[\theta_{(1)}-\theta_{(2)} \mid \theta_{(1)}<\epsilon/6]\cdot\mathbb{P}(\theta_{(1)}<\epsilon/6)+\mathbb{E}[\theta_{(1)}-\theta_{(2)} \mid \theta_{(1)}\geq\epsilon/6]\cdot\mathbb{P}(\theta_{(1)}\geq\epsilon/6)\\ &\leq \epsilon/6+\mathbb{E}[\theta_{(1)}-\theta_{(2)} \mid \theta_{(1)}\geq\epsilon/6]\\ &\leq \epsilon/6+\mathbb{P}(\theta_{(1)}-\theta_{(2)}<\epsilon/6 \mid \theta_{(1)}\geq\epsilon/6)\cdot\epsilon/6+\mathbb{P}(\theta_{(1)}-\theta_{(2)}\geq\epsilon/6 \mid \theta_{(1)}\geq\epsilon/6)\cdot 1\\ &\leq \epsilon/6+\epsilon/6+\epsilon/6\end{aligned}$$

which implies that expected producer surplus is upper bounded by $\epsilon/2$ for all $n \geq \bar{n}_1$. It remains to obtain a lower bound on the total gains from trade, TS:

$$\begin{aligned}TS = \mathbb{E}[\theta_{(1)}] &\geq \mathbb{P}(\theta_{(1)}\geq 1-\delta)\cdot(1-\delta)\\ &= \underbrace{\Big(1-(G(1-\delta))^n\Big)}_{\to 1 \text{ as } n\to+\infty}\cdot(1-\delta)\end{aligned}$$

and so pick $\bar{n}_{2,\epsilon}$ and so that for all $n \geq \bar{n}_{2,\epsilon}$, $TS \geq \epsilon/2$.

Finally, under the matching scheme, each type has her K favourite firms in her consideration set, and we give full information to all firms, this also brings about the efficient outcome which implies that consumer surplus is just total surplus less producer surplus:

$$CS = TS - PS \geq (1-\epsilon/2)-\epsilon/2 = 1-\epsilon$$

for all $n \geq n_\epsilon^{CO} := \max(\bar{n}_{1,\epsilon}, \bar{n}_{2,\epsilon})$ as required. □

We now conclude the proof of Proposition 2. From Lemma 3, there exists n^{PO} so that for all $n \geq n^{PO}$, the producer-optimal point is implemented exactly. From Lemma 4, for any ϵ, there exists n_ϵ^{NT} so that for all $n \geq n_\epsilon^{NT}$, $CS \leq \epsilon$, $PS \leq \epsilon$. From Lemma 5. there exists n_ϵ^{CO} so that for all $n \geq n_\epsilon^{CO}$, $CS \geq 1-\epsilon$ and $PS \leq \epsilon$. Observe that since CS is lower-bounded by zero, this implies that the lower envelope must be approximately linear. Finally, SUR_K is convex. This implies that for all $n \geq \bar{n} := \max\{n^{PO}, n_\delta^{NT}, n_\delta^{CO}\}$ for an appropriately chosen δ (which depends on ϵ),

$$l\Big(SUR \setminus SUR_K\Big) \leq l\Big(PO-NT-CO \setminus SUR_K\Big) \leq \epsilon$$

as required, where $PO-NT-CO$ is the triangle connecting the points $PO=(1,0)$, $NT=(0,0)$, and $CO=(0,1)$.[10]

□

[10] Since the gap between SUR_K and $PPO-NT-PCO$ is upper-bounded by distance δ on all three edges of the triangle, picking $\delta=\epsilon/4$ would suffice.

Supplementary material to "Matching and Information Design in Marketplaces"

Not for publication

Lemma 1 uses the characterization of Theorem 2^C in the online Appendix of Elliott, Galeotti, Koh and Li (2021). For easy reference for a referee, we replicate here the proof of Theorem 2^C in the online Appendix of Elliott, Galeotti, Koh and Li (2021).

Theorem 2^C **(EGKL, 2021).** The consumer-optimal information structure takes the following form: For each $i \in \mathcal{N}$, we apply a uniform profit preserving extremal segmentation according to the distribution of residual valuations and give this as public information.[11]

Proof. As before, we begin by characterizing a lower bound on the profits of firm $i \in \mathcal{N}$. For the same reason as in the main text, this is given by

$$\Pi_i^{C*}(\mathbf{0}) = \max_{p_i \in [0,1]} p_i \int_{\boldsymbol{\theta} \in [0,1]^n : \theta_i - \max_{j \neq i} \theta_j \geq p_i} f(\boldsymbol{\theta}') d\boldsymbol{\theta}'.$$

This is the lowest profit that i can make in any equilibrium induced by any information structure. Denote one of the optimal price of the above problem by p_i^*.

We are once again interested in the distribution of residual valuations—the maximum amount consumers are willing to pay for i's product given that they face prices 0 for all firms $j \neq i$—among consumers in E_i. Define i's effective demand function as

$$D_i^{\text{eff}}(\hat{p}_i) = \int_{\boldsymbol{\theta} \in E_i : \theta_i - \max_{j \neq i} \theta_j \geq p_i} f(\boldsymbol{\theta}') d\boldsymbol{\theta}' \quad \text{for all } \hat{p}_i \in [0,1]$$

which gives the total demand for i's product if for all consumers in E_i, i sets price $\hat{p}_i$ and all other firms $j \neq i$ set price 0. Observe that $D_i^{\text{eff}}(0) - D_i^{\text{eff}}$ is simply a renormalized right-continuous distribution function and so corresponds to a unique measure x_i^{eff} on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ which fulfils $x_i^{eff}([p, +\infty)) = D_i^{\text{eff}}(p)$ for all p.[12]

Now normalizing without loss so that $x_i^{eff}([0,1]) = 1$, Theorem 1B of BBM shows that for any distribution of residual valuations, there exists a $\sigma_i^{\text{eff}} \in \Delta\Delta[0,1]$ such that

$$\int_{x \in \Delta[0,1]} x(B) \sigma_i^{\text{eff}}(dx) = x_i^{\text{eff}}(B) \quad \text{for all Borel sets } B \in \mathcal{B}([0,1])$$

which has the **extremal and uniform profit preserving** property: for each distribution

[11]Explicitly, we can associate each segment with a unique message.
[12]Uniqueness follows from the $\pi - \lambda$ Theorem.

$y \in \text{supp}(\sigma_i^{\text{eff}})$,

$$p_i^* \in \text{supp}(y) = \underset{p \in [0,1]}{\text{argmax}}\, p * y(\text{supp}(y) \cap [p, 1]).$$

If the information designer further segments each E_i by σ_i^{eff}, and sends each segment $y \in \text{supp}(\sigma_i^{\text{eff}})$ to all firms publicly, then firm i charging $\min \text{supp}(y)$ and other firms charge zero is an equilibrium. In this equilibrium, each firm i's profit is driven down to $\Pi_i^C(\mathbf{0})$. But since the allocation is also efficient, this achieves the upper bound of consumer surplus:

$$CS^C = S^C - \sum_{i=1}^{n} \Pi_i^C(\mathbf{0})$$

where

$$S^C = \sum_{i=1}^{n} \int_{\boldsymbol{\theta} \in E_i} f(\boldsymbol{\theta})\theta_i d\boldsymbol{\theta}$$

is the total surplus available.

□